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(The two-dimensional case)

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The Matrizants of the Keplerian Motions

(The two-dimensional case)

by

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In Table II,

instead of $b_{31} = -\frac{\mu}{2HG}\cdots$, read $b_{31} = -\frac{\mu}{2HG^2}$;

$b_{41} = -\frac{\mu}{2HG}\cdots$, read $b_{41} = -\frac{\mu}{2HG^2}$;

$b_{23} = \frac{1}{G}\cdots$, read $b_{23} = -\frac{1}{G}\cdots$;

$b_{33} = \frac{X}{2HG^2}\cdots$, read $b_{33} = -\frac{X}{2HG^2}\cdots$;

$b_{24} = \frac{1}{G}\cdots$, read $b_{24} = -\frac{1}{G}\cdots$;

$b_{34} = \frac{Y}{2HG^2}\cdots$, read $b_{34} = -\frac{Y}{2HG^2}\cdots$.

In Table III,

instead of $a'_{33} = G\frac{Y}{r}$, read $a'_{33} = -G\frac{Y}{r}$.

In Table IV,

instead of $b'_{23} = \frac{1}{G}\cdots$, read $b'_{23} = -\frac{1}{G}\cdots$;

$b'_{24} = -3\frac{t}{r}$, read $b'_{24} = +3\frac{t}{r}$;

$b'_{33} = \frac{\dot{r}}{2HG^2}\cdots$, read $b'_{33} = -\frac{\dot{r}}{2HG^2}\cdots$;

$b'_{34} = \frac{1}{2HGr}\cdots$, read $b'_{34} = -\frac{1}{2HGr}\cdots$.

In Table VI,

instead of $b''_{23} = \frac{2r\dot{r}}{GV} - 3\frac{V}{G}t$, read $b''_{23} = -\frac{2r\dot{r}}{GV} + 3\frac{V}{G}t$,

$b''_{24} = -\frac{2}{V}$, read $b''_{24} = \frac{2}{V}$,

$b''_{33} = \frac{V}{2HG^2}\cdots$, read $b''_{33} = -\frac{V}{2HG^2}\cdots$.

Classically the variational equations for the problem of two bodies have been studied in Jacobi's elliptic elements or Delaunay's canonical variables which make them trivial to solve. This drastic simplification is achieved by successive canonical transformations. Only recently astronomers considered using rectangular coordinates in Perturbation Theory, which led them to compute the partial derivatives of the coordinates along a Keplerian motion with respect to Jacobi's elliptic elements (Eckert and Brouwer 1937, Brouwer and Clemence 1961). From the Eckert-Brouwer formula for differential corrections of Cartesian coordinates, Danby (1965) derived a decomposition of the matrizant $R(t;t_0)$ into the matrix product $R(t;t_0) = M(t) \circ M^{-1}(t_0)$: the matrix $M(t)$ is surprisingly simple in appearance, but no explicit analytical expression is given for its inverse.

Similar to the Eckert-Bronwer formula is Bower's approach to the construction of the matrizant, (Bower 1932). It proceeds in two steps: the variations of the elliptical elements with respect to the initial conditions are first computed, then the variations of the coordinates with respect to the elliptical elements. The algorithm is left in the form of a sequence of operations to be carried on a desk calculator to arrive at the value of the matrizant at a given instant. Following Bower's method, Danby (1962,1965) derived explicit expressions for the elements of the matrizant in the *apsidal* frame of reference (the axis Oz is normal to the orbital plane, while Ox points toward the periastron). Such formulas had already been produced by Myachin (1959) apparently without much explanation as to how he obtained them. The matrizant in the apsidal coordinate system looks slightly simpler when the initial time t_0 is taken at a passage at periastron (Danby 1964).

Starting from the series f and g, Sconzo (1963) arrives at a completely explicit expression of the matrizant but, in view of the forms we have given to it, it is felt that Sconzo's formulas could be presented in a more direct and simpler form.

Brumberg (1961) chooses to operate in the *orbital* coordinate system (the axis O_x points towards the instantaneous position). He is as laconic as Myachin about the way he derives a fundamental set of solutions to the variational equations in this moving frame of reference. Charnyi (1963) made a first attempt at clarifying this issue; lately he has come with a rule to construct a fundamental set of solutions to the adjoint variational equations for a conservative Hamiltonian system with n degrees of freedom when $(n-1)$ independent integrals are known (Charnyi 1965). The author is apparently unaware of the relation between his method and Jacobi's dual theorem of the last multiplier.

To conclude, in treating the variational equations of the problem of two bodies, it is felt that a constructive and systematic approach is needed which would permit an easy transformation from one frame of reference to another and would also facilitate a proper choice of elements under different circumstances.

In order to have a more organized view and to facilitate the search for new forms of the matrizant, an application of Jacobi's dual theorem of the last multiplier is made to the solution of the adjoint variational equations in an arbitrarily given inertial coordinate system. This method permits the formation of the matrizant in a straightforward manner, and its transformation to any other coordinate system such as the orbital frame considered by Brumberg or the *intrinsic* frame postulated by Hill's equation.

Only the two-dimensional problem of two bodies is examined in this paper. Moreover we make quite apparent the places where the treatment we present exclude successively the rectilinear solutions, then the circular motions and finally the parabolic orbits. These restrictions do not come from the introduction of a new independent variable such as the eccentric anomaly and its relation to the time by means of Kepler's equation; rather they follow from the conditions under which the three variational integrals are independent and the fourth solution associated with the last multiplier is independent from the gradients of the variational integrals.

A special effort is made to keep the matrizants in the form most suitable to repeated processing by electronic computers. As far as possible we eschew introducing transcendental functions; the time is kept as the running variable along the orbit so that there is no need for solving Kepler's equation. We hope that formulas of this kind would prove to be elementary enough to fit into an airborne computer, when they are extended to the three dimensional problem.

1. Jacobi's theorem of the last multiplier applied to the variational equations.

As it is usually done, we use the conservation of linear momentum to reduce the problem of two bodies to a dynamical system described by the Hamiltonian function

$$\mathcal{H} = \frac{1}{2}(X^2 + Y^2) - \frac{\mu}{r} \tag{1}$$

in a four dimensional phase space product of the Cartesian plane of configurations (x,y) by the Cartesian plane of moments (X,Y). In the function (1), μ is a strictly positive real constant and

$$r = |x^2 + y^2|^{\frac{1}{2}} \tag{2}$$

The Hamiltonian equations generated from (1) will be called *orbital*. They admit four integrals:

(i) the energy

$$H = \frac{1}{2}(X^2 + Y^2) - \frac{\mu}{r}\ ; \tag{3}$$

(ii) the angular momentum

$$G = xY - yX; \tag{4}$$

(iii) Laplace's vector given by its Cartesisn components

$$P = \quad GY - \mu\frac{x}{r} \tag{5}$$

$$Q = -\ GX - \mu\frac{y}{r} \tag{6}$$

The functions (3)-(6) will be termed *orbital integrals*. We agree to denote by the same symbol either the function which is an integral or the constant value which this function takes along a particular solution of the orbital equations; no confusion could result from this ambiguity in the notations.

We recall an essential formula relating the four orbital integrals, namely

$$P^2 + Q^2 = \mu^2 + 2HG^2, \tag{7}$$

and we set up a list of basic identities

$$\begin{aligned} Px + Qy &= G^2 - \mu r, \\ Py - Qx &= Gr\dot{r}, \\ PX + QY &= -\mu\dot{r}, \\ PY - QX &= G(V^2 - \frac{\mu}{r}) \end{aligned} \tag{8}$$

which we shall use quite often in the following sections of this paper without explicitly referring the reader to them.

Let

$$\Gamma: t \to (x(t), y(t), X(t), Y(t)) \tag{9}$$

be a solution of the orbital equations. By definition a displacement or variation of Γ is any solution $\underset{\sim}{u} = (u,w,U,W)$ of the so-called variational equations

$$\begin{aligned} \dot{u} &= U \qquad & \dot{U} &= \frac{\mu}{r^3}\left[\left(3\frac{x^2}{r^2} - 1\right) u + 3\frac{xy}{r^2}w\right] \\ \dot{w} &= W \qquad & \dot{W} &= \frac{\mu}{r^3}\left[3\frac{xy}{r^2}u + \left(3\frac{y^2}{r^2} - 1\right)w\right] \end{aligned} \tag{10}$$

considered along the orbit Γ.

The variational equations are Hamiltonian: they are generated from the Hamiltonian function

$$\mathcal{V} = \frac{1}{2}(U^2 + W^2) - \frac{1}{2}\frac{\mu}{r^3}\left[\left(3\frac{x^2}{r^2} - 1\right)u^2 + 6\frac{xy}{r^2}\,uw + \left(3\frac{y^2}{r^2} - 1\right)w^2\right], \tag{11}$$

which, as it is expected, is a quadratic form in the vector $\underset{\sim}{u}$ whose coefficients are functions of the time to be computed along the orbit (9).

Each orbital integral F generates an integral δF of the variational equations, which we shall call here a *variational integral.* It is to be built as the scalar product $\delta F = (\underset{\sim}{u}|\nabla F)$ of the vector $\underset{\sim}{u}$ by the gradient $\nabla F = (F_x, F_y, F_X, F_Y)$ of the function F in the phase space (x, y, X, Y).

Let us compute the gradients ∇G, ∇P and ∇Q of the orbital integrals (4), (5) and (6):

$$
\begin{aligned}
G_x &= Y, & P_x &= Y^2 - \mu \frac{y^2}{r^3} & Q_x &= -XY + \mu \frac{xy}{r^3} \\
G_y &= -X, & P_y &= -XY + \mu \frac{xy}{r^3} & Q_y &= X^2 - \mu \frac{x^2}{r^3} \\
G_X &= -y, & P_X &= -yY & Q_X &= 2yX - xY \\
G_Y &= x, & P_Y &= 2xY - yX & Q_Y &= -xX.
\end{aligned}
\tag{12}
$$

We shall now build the six-dimensional 2-vector $\nabla P \wedge \nabla Q$ which is the exterior product of the vector

$$\nabla P = P_x \mathbf{e}_1 + P_y \mathbf{e}_2 + P_X \mathbf{e}_3 + P_Y \mathbf{e}_4$$

by the vector

$$\nabla Q = Q_x \mathbf{e}_1 + Q_y \mathbf{e}_2 + Q_X \mathbf{e}_3 + Q_Y \mathbf{e}_4.$$

The components of the result written as the sum

$$\nabla P \wedge \nabla Q = \sum_{(i,j)} A_{ij} \, \mathbf{e}_i \wedge \mathbf{e}_j$$

are found to be

$$
\begin{aligned}
A_{12} &= -\frac{\mu G^2}{r^3} & A_{23} &= G\left(XY - 2\mu \frac{xy}{r^3}\right) \\
A_{13} &= -G\left(Y^2 - 2\mu \frac{y^2}{r^3}\right) & A_{24} &= -G\left(X^2 - 2\mu \frac{x^2}{r^3}\right) \\
A_{14} &= G\left(XY - 2\mu \frac{xy}{r^3}\right) & A_{34} &= 2G^2.
\end{aligned}
\tag{13}
$$

In the same manner we construct the exterior products

$$\nabla G \wedge \nabla P = \sum_{(i,j)} B_{ij}\, e_i \wedge e_j ,$$

$$\nabla G \wedge \nabla Q = \sum_{(i,j)} C_{ij}\, e_i \wedge e_j$$

whose components are as follows:

$$\begin{aligned}
B_{12} &= -\mu G \frac{y}{r^3} & C_{12} &= -\mu G \frac{x}{r^3} \\
B_{13} &= -\mu \frac{y^3}{r^3} & C_{13} &= -GY + \mu \frac{xy^2}{r^3} \\
B_{14} &= GY + \mu \frac{xy^2}{r^3} & C_{14} &= -\mu \frac{x^2 y}{r^3} \\
B_{23} &= \mu \frac{xy^2}{r^3} & C_{23} &= GX - \mu \frac{x^2 y}{r^3} \\
B_{24} &= -GX - \mu \frac{x^2 y}{r^3} & C_{24} &= \mu \frac{x^3}{r^3} \\
B_{34} &= -Gy & C_{34} &= Gx.
\end{aligned} \tag{14}$$

The following two properties are equivalent:

a) The gradients ∇P, ∇Q *and* ∇G *are linearly independent;*

b) The Keplerian motion Γ *is not rectilinear.*

Indeed let us build the four-dimensional 3-vector

$$\nabla G \wedge \nabla P \wedge \nabla Q = \sum_{(i,j,k)} D_{ijk}\, e_i \wedge e_j \wedge e_k .$$

Its coefficients are as follows

$$\begin{aligned}
D_{123} &= -\mu G^2 \frac{y}{r^3} & D_{134} &= G^2 Y, \\
D_{124} &= \mu G^2 \frac{x}{r^3} & D_{234} &= -G^2 X.
\end{aligned} \tag{15}$$

But the gradients ∇P, ∇Q and ∇G are linearly dependent if and only if the 3-vector $\nabla G \wedge \nabla P \wedge \nabla Q$ vanishes identically. As it is exhibited by the components D_{ijk} found in (15), this is the case if and only if the angular momentum G is zero, thus if and only if the Keplerian motion is rectilinear.

From now on in this paper, we shall assume that G is a nonvanishing constant; provided the plane (x,y) of configurations is given the good orientation, we can even assume that G is strictly positive.

Since G is $\neq 0$, the fundamental identity (7) implies that

$$\nabla H = -\frac{2H}{G}\nabla G + \frac{P}{G^2}\nabla P + \frac{Q}{G^2}\nabla Q \tag{16}$$

which means that the gradient ∇H of the orbital energy belongs to the three-dimensional vector subspace generated by the gradients ∇G, ∇P and ∇Q. Consequently, the orbital integrals produce only three linearly independent integrals, namely the functions:

$$\begin{aligned}
\delta G &= (\underset{\sim}{u}|\nabla G) = G_x u + G_y w + G_X U + G_Y W,\\
\delta P &= (\underset{\sim}{u}|\nabla P) = P_x u + P_y w + P_X U + P_Y W,\\
\delta Q &= (\underset{\sim}{u}|\nabla Q) = Q_x u + Q_y w + Q_X U + Q_Y W.
\end{aligned} \tag{17}$$

Now, because they are Hamiltonian, the variational equations have a multiplier, which is equal to 1. Accordingly, in view of Jacobi's theorem of the last multiplier, the variational equations are to be solved by quadratures only.

2. Resolution of the adjoint variational equations.

In this paragraph we set ourselves upon the task of bringing out the quadratures upon which hinges the solution of the variational equations.

$$\dot{u}^* = -\frac{\mu}{r^3}\left[\left(3\frac{x^2}{r^2}-1\right)U^* + 3\frac{xy}{r^2}W^*\right], \qquad \dot{U}^* = -u^*,$$
$$\dot{w}^* = -\frac{\mu}{r^3}\left[3\frac{xy}{r^2}U^* + \left(3\frac{y^2}{r^2}-1\right)W^*\right], \qquad \dot{W}^* = -w^*. \tag{17}$$

Like the variational equations, the adjoint variational equations are Hamiltonian; indeed they are generated by the Hamiltonian function

$$\mathcal{V}^* = -\frac{1}{2}\frac{\mu}{r^3}\left[\left(3\frac{x^2}{r^2}-1\right)U^{*2} + 6\frac{xy}{r^2}U^*W^* + \left(3\frac{y^2}{r^2}-1\right)W^{*2}\right] + \frac{1}{2}(u^{*2}+w^{*2}) \tag{18}$$

Notice how one can go from the variational system (10) to its adjoint by the completely canonical mapping

$$u = U^*, \quad w = W^*, \qquad U = -u^*, \quad W = -w^*. \tag{19}$$

This is a nontrivial example of the law of exchange between coordinates and momenta in a phase space.

As it is well known, any orbital integral gives rise by means of its gradient to a solution of the adjoint variational equations. Thus, in view of the proposition we proved in the preceding paragraph, the gradients ∇G, ∇P, ∇Q constitute three linearly independent solutions of the equations (17). Now we refer ourselves to the dual of Jacobi's theorem of the last multiplier: the adjoint equations (17) having a multiplier (which is equal to 1), once we

know three independent solutions, we must be able to produce the general solution by means of quadratures only.

To this effect let us introduce four unknown functions $\alpha,\beta,\gamma,\delta$ of the time under the provision that the linear combinations

$$\begin{aligned} u^* &= \alpha G_x + \beta P_x + \gamma Q_x, \\ w^* &= \alpha G_y + \beta P_y + \gamma Q_y, \\ U^* &= \alpha G_X + \beta P_X + \gamma Q_X + \delta, \\ W^* &= \alpha G_Y + \beta P_Y + \gamma Q_Y \end{aligned} \tag{20}$$

make a solution of the adjoint variational equations (17). On substituting the expressions (20) into the equations (17), we find that the unknowns α,β,γ and δ ought to be solutions of the differential system

$$\dot\alpha G_x + \dot\beta P_x + \dot\gamma Q_x = -\frac{\mu}{r^3}\left(3\frac{x^2}{r^2} - 1\right)\delta, \tag{21a}$$

$$\dot\alpha G_y + \dot\beta P_y + \dot\gamma Q_y = -3\frac{\mu}{r^3}\frac{xy}{r^2}\delta, \tag{21b}$$

$$\dot\alpha G_X + \dot\beta P_X + \dot\gamma Q_X + \dot\delta = 0, \tag{21c}$$

$$\dot\alpha G_Y + \dot\beta P_Y + \dot\gamma Q_Y = 0. \tag{21d}$$

In order to solve it, we begin by using the equations (21a) and (21b) to express $\dot\beta$ and $\dot\gamma$ by means of $\dot\alpha$ and δ, so that

$$A_{12}\dot\beta = \frac{\mu}{r^3}\left[3\frac{xy}{r^2}Q_x - \left(3\frac{x^2}{r^2} - 1\right)Q_y\right]\delta - C_{12}\dot\alpha, \tag{22a}$$

$$A_{12}\dot\gamma = \frac{\mu}{r^3}\left[3\frac{xy}{r^2}P_x + \left(3\frac{x^2}{r^2} - 1\right)P_y\right]\delta + B_{12}\dot\alpha; \tag{22b}$$

we bring these expressions into the equation (21d) to express $\dot{\alpha}$ in terms of δ, which gives the relation

$$D_{124}\dot{\alpha} = \frac{\mu}{r^3}\left[3\frac{xy}{r^2}A_{14} - \left(3\frac{x^2}{r^2} - 1\right)A_{24}\right]\delta. \tag{22c}$$

But we observe that

$$3\frac{xy}{r^2}A_{14} - \left(3\frac{x^2}{r^2} - 1\right)A_{24} = G\frac{x^2}{r^3}\left[\frac{d}{dt}\left(\frac{r^3\dot{x}}{x}\right) - 3\mu\right],$$

so that (22c) becomes

$$\dot{\alpha} = \frac{x\delta}{Gr^3}\left[\frac{d}{dt}\left(\frac{r^3\dot{x}}{x}\right) - 3\mu\right]. \tag{23a}$$

We then substitute (23a) into (22a) and (22b) to obtain the relations

$$\dot{\beta} = \frac{2\mu x^2\delta}{G^2r^3}, \tag{23b}$$

$$\dot{\gamma} = \frac{x\delta}{G^2r^3}\left[2\mu y + G\frac{d}{dt}\left(\frac{r^3}{x}\right)\right]. \tag{23c}$$

On replacing $\dot{\alpha}$, $\dot{\beta}$ and $\dot{\gamma}$ in (21c) by their expressions as given by the relations (23), we transform (21c) into the differential equation

$$A_{12}D_{124}\dot{\delta} = -\frac{\mu\delta}{r^3}\left[\left(3\frac{x^2}{r^2} - 1\right)(A_{23}D_{124}-A_{24}D_{123}) + 3\frac{xy}{r^2}(A_{14}D_{123}-A_{13}D_{124})\right].$$

But from the definitions (13) and (15), we compute that

$$A_{23}D_{124}-A_{24}D_{123} = \frac{\mu G^4}{r^3}X,$$

$$A_{14}D_{123}-A_{13}D_{124} = \frac{\mu G^4}{r^3}Y.$$

Accordingly the linear differential equations which define δ take the following simple form

$$\dot{\delta} = \frac{x\delta}{r^3}\frac{d}{dt}\left(\frac{x}{r^3}\right);$$

it is immediately integrable by quadrature to give that

$$\delta = K_1 G \frac{r^3}{x}, \tag{24a}$$

K_1 being an arbitrary constant of integration. When we substitute this expression of δ into the equations (23) and perform the quadratures as indicated, we obtain that

$$\alpha = K_2 - K_1\left(3\mu t - \frac{r^3 X}{x}\right), \tag{24b}$$

$$\beta = K_3 + K_1\frac{2\mu}{G}\int x dt, \tag{24c}$$

$$\gamma = K_4 + K_1\frac{2\mu}{G}\int y dt + K_1\frac{r^3}{x}. \tag{24d}$$

This concludes the task we assigned to ourselves in this paragraph: in conformity with the dual of Jacobi's theorem of the last multiplier, solving the adjoint variational equations (17) reduces to performing the quadratures indicated in (24c) and (24d).

3. The quadratures required to solve the adjoint variational equations.

By going back to the first two identities (8), we immediately derive that

$$\begin{aligned}\int x\ dt &= \frac{1}{P^2+Q^2}\left[G(GPt - \tfrac{1}{2}Qr^2) - \mu P\int r\ dt\right],\\ \int y\ dt &= \frac{1}{P^2+Q^2}\left[G(GQt + \tfrac{1}{2}Pr^2) - \mu Q\int r\ dt\right],\end{aligned} \tag{25}$$

provided of course that $P^2 + Q^2 \neq 0$, which is the case if and only if the Keplerian motion (9) is not circular. This will be our assumption in the present paragraph. In view of (25), the problem is now reduced to constructing a primitive of r.

Let us begin by observing that

$$\int r\,dt = \int \dot{r}\left(\frac{r}{\dot{r}}\right)dt = \frac{r^2}{\dot{r}} - \int r\,\frac{d}{dt}\left(\frac{r}{\dot{r}}\right)$$

or else, in view of the equation for the radial motion

$$\ddot{r} = \frac{G^2}{r^3} - \frac{\mu}{r^2}, \tag{26}$$

that

$$2\int r\,dt = \left(\frac{r}{\dot{r}}\right)^2 + G^2\int\frac{dt}{\dot{r}^2 r} - \mu\int\frac{dt}{\dot{r}^2}. \tag{27}$$

But Charnyi (1963) has shown how the two primitives contained in the right member of (27) are to be constructed.

For convenience we put

$$I_n = \int\frac{dt}{\dot{r}^2 r^n} \qquad (n \geq 0). \tag{28}$$

On dividing the orbital integral of energy written in the form

$$H = \tfrac{1}{2}\left(\dot{r}^2 + \frac{G^2}{r^2}\right) - \frac{\mu}{r} \tag{29}$$

first by $\dot{r}^2$ and then by $\dot{r}^2 r$, we find that

$$2HI_0 + 2\mu I_1 - G^2 I_2 = t, \tag{30a}$$

$$2HI_1 + 2\mu I_2 - G^2 I_3 = \int\frac{dt}{r}. \tag{30b}$$

In the same way, on dividing the equation (26) first by $\dot{r}^2$ and then by $\dot{r}^2/r$, we obtain that

$$\mu I_1 - G^2 I_2 = -\int \frac{r\ddot{r}}{\dot{r}^2}\,dt, \tag{30c}$$

$$\mu I_2 - G^2 I_3 = -\int \frac{\ddot{r}}{\dot{r}^2}\,dt. \tag{30d}$$

However, on account of (26), an integration by parts yields immediately that

$$\int \dot{r}^2 dt = r\dot{r} - \int\left(\frac{G}{r^2} - \frac{\mu}{r}\right)dt,$$

whence, by recourse to the integral (29), we establish that

$$\int \frac{dt}{r} = \frac{1}{\mu}\,(r\dot{r} - 2Ht); \tag{31a}$$

this exhibits the right member of (30b) in a closed form. On the other hand we observe that

$$\frac{d}{dt}\left(\frac{r}{\dot{r}}\right) = 1 - \frac{r\ddot{r}}{\dot{r}^2}$$

wherefrom, by quadrature, we arrive at the primitive

$$\int \frac{r\ddot{r}}{\dot{r}^2}\,dt = t - \frac{r}{\dot{r}}\;; \tag{31b}$$

this produces the right member of (30c) in a closed form. Eventually, because

$$\frac{d}{dt}\left(\frac{1}{\dot{r}}\right) = -\frac{\ddot{r}}{\dot{r}^2}\,,$$

we find that

$$\int \frac{\ddot{r}}{\dot{r}^2}\,dt = -\frac{1}{\dot{r}}\,, \tag{31c}$$

and so we exhibit the right member of (30d) in a closed form. To conclude:

in order to construct the primitives I_0 and I_1 requested by (27), all we have to do is to solve the linear system (30) in the unknowns $I_n (n = 0,1,2,3)$, provided we use the primitives (31).

We use the equations (30c) and (30d) to express I_1 and I_3 as functions of I_2; then we substitute the results into (30b) and we find that

$$I_2 = \frac{\mu r - G^2}{(P^2+Q^2) r\dot{r}} \tag{32a}$$

Thereafter we go back to the equation (30c) to obtain that

$$I_1 = \frac{1}{\mu\dot{r}}\left[r + \frac{G^2}{P^2+Q^2}\left(\mu - \frac{G^2}{r}\right)\right] - \frac{1}{\mu}t \; ; \tag{32b}$$

on substituting (32a) and (32b) into (30a), we find that

$$I_0 = \frac{3t}{2H} - \frac{1}{2H\dot{r}}\left[2r + \frac{G^2}{P^2+Q^2}\left(\mu - \frac{G^2}{r}\right)\right]. \tag{32c}$$

At this point, in order to divide both sides of (32c) by H, we have to exclude the parabolic motions, which we shall do for the remainder of this paragraph and the next one.

Let us replace I_0 and I_1 in (27) by their expressions as given by (32b) and (32c). After some cumbersome but elementary transformations, we finally find that

$$\int r \, dt = \frac{r\dot{r}}{4\mu H}(G^2 + \mu r) - \tfrac{1}{2}\left(\frac{3\mu}{2H} + \frac{G^2}{\mu}\right)t. \tag{33}$$

It follows therefrom by the relations (25) tha

$$\int x \, dt = \frac{3P}{4H}t - r\,\frac{P\dot{r}(\mu r + G^2) + 2HGQr}{4H(P^2 + Q^2)} \tag{34a}$$

$$\int y \, dt = \frac{3Q}{4H}t - r\,\frac{Q\dot{r}(\mu r + G^2) - 2HGPr}{4H(P^2 + Q^2)} \tag{34b}$$

After a few more manipulations we arrive at a simpler form for these primitives:

$$\int x \, dt = \frac{3P}{4H}t + \frac{1}{4H}(Xr^2 - Gy), \tag{35a}$$

$$\int y \, dt = \frac{3Q}{4H}t + \frac{1}{4H}(Yr^2 + Gx). \tag{35b}$$

It should be said that we obtain the right side of (35a) by omitting the constant $QG^3/4H(P^2+Q^2)$ from the right side of (34a), and the right side of (35b) by likewise omitting the constant $-PG^3/4H(P^2+Q^2)$ from the right side of (34b).

To our knowledge, the formulas (33), (35a) and (35b) appear to be new in the literature of the problem of two bodies.

4. General solution of the variational equations.

Before we compute the general solution of the variational equations for an elliptic or an hyperbolic motion, let us take the partial derivatives of the basic identity (7) so that

$$\begin{aligned} PP_x + QQ_x &= G(\mu G \frac{x}{r^3} + 2HY), \\ PP_y + QQ_y &= G(\mu G \frac{y}{r^3} - 2HX), \\ PP_X + QQ_X &= G(Gx - 2Hy), \\ PP_Y + QQ_Y &= G(Gy + 2Hx). \end{aligned} \tag{36}$$

In view of these formulas we draw from the quadratures (35) that

$$
\begin{aligned}
P_x \int x\, dt + Q_x \int y\, dt &= \frac{G}{4H}\left[3t\left(\mu G \frac{x}{r^3} + 2HY\right) - 2Hy - GX\right], \\
P_y \int x\, dt + Q_y \int y\, dt &= \frac{G}{4H}\left[3t\left(\mu G \frac{y}{r^3} - 2HX\right) + 2Hx - GY\right], \\
P_X \int x\, dt + Q_X \int y\, dt &= \frac{G}{4H}\left[3t\left(GX \quad - 2Hy\right) - 2G^2 x\right], \\
P_Y \int x\, dt + Q_Y \int y\, dt &= \frac{G}{4H}\left[3t\left(GY \quad + 2Hx\right) - 2G^2 y\right].
\end{aligned}
\tag{37}
$$

We are now ready to substitute the functions α, β, γ, δ given by the formulae (24) into the linear combinations (20). A few trivial manipulations yield the final result:

$$
\begin{aligned}
u^* &= K_1'(X - 3\mu \frac{x}{r^3} t) + K_2 G_x + K_3 P_x + K_4 Q_x, \\
v^* &= K_1'(Y - 3\mu \frac{y}{r^3} t) + K_2 G_y + K_3 P_y + K_4 Q_y, \\
U^* &= K_1'(2x - 3Xt) \quad + K_2 G_X + K_3 P_X + K_4 Q_X, \\
V^* &= K_1'(2y - 3Yt) \quad + K_2 G_Y + K_3 P_Y + K_4 Q_Y,
\end{aligned}
\tag{38}
$$

it being understood that the arbitrary constant K_1 of integration is now replaced by a constant K_1' such that $2HK_1' = -\mu GK_1$.

We find it convenient to define the vector

$$\underset{\sim}{S} = S_1 \underset{\sim}{e}_1 + S_2 \underset{\sim}{e}_2 + S_3 \underset{\sim}{e}_3 + S_4 \underset{\sim}{e}_4$$

whose components are

$$S_1 = X - 3\mu \frac{x}{r^3} t, \qquad S_3 = 2x - 3Xt,$$

$$S_2 = Y - 3\mu \frac{y}{r^3} t, \qquad S_4 = 2y - 3Yt.$$

We just proved that $\underset{\sim}{S}$ is a solution of the adjoint variational equations (17).

Let us compute the one-dimensional 4-vector $\underset{\sim}{S} \wedge \nabla G \wedge \nabla P \wedge \nabla Q$; we find that

$$\underset{\sim}{S} \wedge \nabla G \wedge \nabla P \wedge \nabla Q = -2HG^2 \, \underset{\sim}{e}_1 \wedge \underset{\sim}{e}_2 \wedge \underset{\sim}{e}_3 \wedge \underset{\sim}{e}_4.$$

This proves that the matrix

$$A^*(t) = (a^*_{ij}(t))$$

whose columns are the vectors $\underset{\sim}{S}$, ∇G, ∇P, ∇Q taken in that order has its determinant equal to $-2HG^2$; it thus constitutes a fundamental set of solutions to the adjoint variational equations.

In order to compute the inverse $B^*(t)$ of $A^*(t)$, we compute the following 3-vectors:

$$\underset{\sim}{S} \wedge \nabla G \wedge \nabla P = \sum_{(i,j,k)} E_{ijk} \, \underset{\sim}{e}_i \wedge \underset{\sim}{e}_j \wedge \underset{\sim}{e}_k,$$

$$\underset{\sim}{S} \wedge \nabla G \wedge \nabla Q = \sum_{(i,j,k)} F_{ijk} \, \underset{\sim}{e}_i \wedge \underset{\sim}{e}_j \wedge \underset{\sim}{e}_k,$$

$$\underset{\sim}{S} \wedge \nabla P \wedge \nabla Q = \sum_{(i,j,k)} J_{ijk} \, \underset{\sim}{e}_i \wedge \underset{\sim}{e}_j \wedge \underset{\sim}{e}_k,$$

the components are

$$E_{123} = \mu \frac{y}{r^3} (Yr^2 + Gx + 3Qt)$$

$$E_{124} = -\mu\frac{x}{r^3}\,(Yr^2 + Gx + 3Qt) - 2HG,$$

$$E_{134} = -\;Y\;(Yr^2 + Gx + 3Qt) + 2Hy^2,$$

$$E_{234} = \;X\;(Yr^2 + Gx + 3Qt) - 2Hxy,$$

$$F_{123} = -\mu\frac{y}{r^3}\,(Xr^2 - Gy + 3Pt) + 2HG,$$

$$F_{124} = \mu\frac{x}{r^3}\,(Xr^2 - Gy + 3Pt),$$

$$F_{134} = \;Y\;(Xr^2 - Gy + 3Pt) - 2Hxy,$$

$$F_{234} = -\;X\;(Xr^2 - Gy + 3Pt) + 2Hx^2,$$

$$J_{123} = 2HG\,(Y - 3\mu\frac{y}{r^3}t),$$

$$J_{124} = -2HG\,(X - 3\mu\frac{x}{r^3}t),$$

$$J_{134} = -2HG\,(2y - 3Yt),$$

$$J_{234} = 2HG\,(2x - 3Xt).$$

But, as we know from Linear Algebra, these 3-vectors and the one we computed in (17) yield the elements b^*_{ij} of the inverse matrix B^* from the following relations:

$$-2HG^2\cdot b^*_{11} = D_{234}, \qquad -2HG^2\cdot b^*_{21} = -J_{234},$$

$$-2HG^2\cdot b^*_{12} = -D_{134}, \qquad -2HG^2\cdot b^*_{22} = J_{134},$$

$$-2HG^2\cdot b^*_{13} = D_{124}, \qquad -2HG^2\cdot b^*_{23} = -J_{124},$$

$$-2HG^2\cdot b^*_{14} = -D_{123}, \qquad -2HG^2\cdot b^*_{24} = J_{123},$$

$$-2HG^2_{\cdot}\, b^*_{31} = F_{234}, \qquad -2HG^2_{\cdot}\, b^*_{41} = -E_{234},$$

$$-2HG^2_{\cdot}\, b^*_{32} = -F_{134}, \qquad -2HG^2_{\cdot}\, b^*_{42} = E_{134},$$

$$-2HG^2_{\cdot}\, b^*_{33} = F_{124}, \qquad -2HG^2_{\cdot}\, b^*_{43} = -E_{124},$$

$$-2HG^2_{\cdot}\, b^*_{34} = -F_{123}, \qquad -2HG^2_{\cdot}\, b^*_{44} = E_{123}.$$

The matrix $A^*(t)$ being a fundamental set of solutions to the equations (17), the matrix

$$R^*(t;t_0) = A^*(t)\circ B^*(t_0) \tag{39}$$

is also such a set. But obviously

$$R^*(t_0;t_0) = A^*(t_0)\circ B^*(t_0) = 1_4.$$

Hence $R^*(t;t_0)$ is the resolvent of the adjoint variational equations.

Now, in view of (19), the canonical matrix

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

is such that the vector

$$\underset{\sim}{u} = J.\underset{\sim}{u}^*$$

is a variation (i.e. a solution of the equations (10)) if and only if the vector $\underset{\sim}{u}^*$ is an adjoint variation. Therefore the matrix

$$R(t;t_0) = J\circ R^*(t;t_0)\circ J^{-1} \tag{40}$$

is the resolvent of the variational equations. A simple expression of $R(t;t_0)$ is obtained by putting

$$A(t) = J \circ A^{*}(t), \tag{41a}$$

$$B(t) = B^{*}(t) \circ J^{-1} \tag{41b}$$

so that

$$R(t;t_0) = A(t) \circ B(t_0). \tag{42}$$

The elements of the matrices $A(t)$ and $B(t)$ are presented in Tables I and II.

Table I - The factor $A(t)$ of the matrizant $R(t,t_0)$

$a_{11} = 2x - 3Xt$	$a_{12} = -y$
$a_{21} = 2y - 3Yt$	$a_{22} = x$
$a_{31} = -X + 3\mu \frac{x}{r^3} t$	$a_{32} = -Y$
$a_{41} = -Y + 3\mu \frac{y}{r^3} t$	$a_{42} = X$
$a_{13} = -yY$	$a_{14} = -xY + 2yX$
$a_{23} = -yX + 2xY$	$a_{24} = -xX$
$a_{33} = -Y^2 + \mu \frac{y^2}{r^3}$	$a_{34} = XY - \mu \frac{xy}{r^3}$
$a_{43} = XY - \mu \frac{xy}{r^3}$	$a_{44} = -X^2 + \mu \frac{x^2}{r^3}$

Table II - The factor $B(t)$ of the matrizant $R(t;t_0)$

$b_{11} = -\frac{\mu}{2H}\frac{x}{r^3}$	$b_{12} = -\frac{\mu}{2H}\frac{y}{r^3}$
$b_{21} = \frac{1}{G}(-X + 3\mu\frac{x}{r^3}t)$	$b_{22} = \frac{1}{G}(-Y + 3\mu\frac{y}{r^3}t)$
$b_{31} = -\frac{\mu}{2HG}\frac{x}{r^3}(Xr^2 - Gy + 3Pt)$	$b_{32} = -\frac{\mu}{2HG^2}\frac{y}{r^3}(Xr^2 - Gy + 3Pt) + \frac{1}{G}$
$b_{41} = -\frac{\mu}{2HG}\frac{x}{r^3}(Yr^2 + Gx + 3Qt) - \frac{1}{G}$	$b_{42} = -\frac{\mu}{2HG^2}\frac{y}{r^3}(Yr^2 + Gx + 3Qt)$
$b_{13} = -\frac{X}{2H}$	$b_{14} = -\frac{Y}{2H}$
$b_{23} = \frac{1}{G}(2x - 3Xt)$	$b_{24} = \frac{1}{G}(2y - 3Yt)$
$b_{33} = \frac{X}{2HG^2}(Xr^2 - Gy + 3Pt) + \frac{x^2}{G^2}$	$b_{34} = \frac{Y}{2HG^2}(Xr^2 - Gy + 3Pt) + \frac{xy}{G^2}$
$b_{43} = -\frac{X}{2HG^2}(Yr^2 + Gx + 3Qt) + \frac{xy}{G^2}$	$b_{44} = -\frac{Y}{2HG^2}(Yr^2 + Gx + 3Qt) + \frac{y^2}{G^2}$

5. The matrizant in the orbital frame of reference.

The resolvent (42) is related to inertial Cartesian frames of reference in the original space: given a displacement $\underset{\sim}{\Delta}_0 = (\Delta x_0, \Delta y_0, \Delta\dot{x}_0, \Delta\dot{y}_0)$ of the initial position (x_0, y_0) and of the initial velocity $(\dot{x}_0, \dot{y}_0)$ with respect to that fixed frame, it provides in the linear approximation the evolution $\underset{\sim}{\Delta}(t) = (\Delta x(t), \Delta y(t), \Delta\dot{x}(t), \Delta\dot{y}(t))$ of that displacement along the orbit Γ, namely

$$\underset{\sim}{\Delta}(t) = R(t;t_0).\Delta_0.$$

But there are other frames of reference, not fixed in space, to which it is sometimes more convenient in practice to refer the Keplerian orbit. In this paragraph, we shall consider the *orbital* frame of reference: the origin O is the center of attraction, the axis Ox' is the line from O to the position occupied by the particle, and the axis Oy' is taken orthogonally to the axis Ox' and 90° forward of the axis Ox' in the sense of the motion of the particle on its orbit.

If θ is the azimuth of the particle in the inertial frame of reference, such that

$$x = r\cos\theta, \qquad y = r\sin\theta, \tag{43}$$

then its Cartesian coordinates in the orbital frame $Ox'y'$ are

$$x' = r, \qquad y' = 0,$$

and the Cartesian components in the frame $Ox'y'$ of its velocity with respect to the frame Oxy are

$$X' = \dot{r}, \qquad Y' = r\dot{\theta} = \frac{G}{r}. \tag{44}$$

Let (u', w', U', W') be the components in the frame $Ox'y'$ of the displacement $\underset{\sim}{u}$ in the frame Oxy; obviously

$$\begin{aligned}
u &= u'\cos\theta + w'\sin\theta,\\
w &= -u'\sin\theta + w'\cos\theta,\\
U &= U'\cos\theta + W'\sin\theta,\\
W &= -U'\sin\theta + W'\cos\theta.
\end{aligned} \tag{45}$$

Thus the components u', w', U', W' of the displacement can be interpreted as follows:

(i) u' is the correction in radial position or range;

(ii) w' the correction in transversal position;

(iii) U' the correction in radial velocity or range rate;

(iv) W' the correction in transversal velocity.

Notice that the relations (45) define a time-dependent canonical tranformation; it can be taken as being generated by the function

$$\mathcal{S}' \equiv \mathcal{S}'(U,W,u',w') = -U(u' \cos\theta + w' \sin\theta) - W(-u' \sin\theta + w' \cos\theta)$$

so that its remainder is the function

$$\frac{\partial \mathcal{S}'}{\partial t} = -\ (uW - wU)\dot{\theta} = -G(uW - wU)\dot{\theta}. \tag{46}$$

The transformed variational Hamiltonian is the function

$$\mathcal{V}' \equiv \mathcal{V}'(u',w',U',W') = \mathcal{V} + \frac{\partial \mathcal{S}'}{\partial t}. \tag{47}$$

Since the transformation (45) is a proper rotation in the plane of configurations and in the plane of moments, we have immediately that

$$U^2 + W^2 = U'^2 + W'^2, \tag{48a}$$

$$uW - wU = u'W' - w'U'; \tag{48b}$$

we also compute that

$$\left(3\frac{x^2}{r^2} - 1\right)u^2 + 6xy\,\frac{uw}{r^2} + \left(3\frac{y^2}{r^2} - 1\right)w^2 = 2u'^2 - w'^2. \tag{48c}$$

This last expression is interesting in that it exhibits that the orbital axes Ox' and Oy' are the principal axes of that quadratic form which is the force function in the variational Hamiltonian $\mathcal{V}$. Thus, mathematically

speaking, the orbital axes Ox' and Oy' are introduced to *diagonalize* partially the variational Hamiltonian.

On introducing (48a), (48b) and (48c) into (11) and (46), we eventually obtain from (47) that

$$\mathcal{V}' = \tfrac{1}{2}(U'^2+W'^2) - \frac{G^2}{r^2}(u'W-w'U') - \frac{\mu}{2r^3}(2u'^2-w'^2) \tag{49}$$

The variational equations derived from $\mathcal{V}'$ are

$$\begin{aligned} \dot{u}' &= \frac{\partial\mathcal{V}'}{\partial U'} = U' + \frac{G}{r^2}w', & \dot{U}' &= -\frac{\partial\mathcal{V}'}{\partial u'} = \frac{G}{r^2}W' + \frac{\mu}{r^3}u', \\ \dot{w}' &= \frac{\partial\mathcal{V}'}{\partial W'} = W' - \frac{G}{r^2}u', & \dot{W}' &= -\frac{\partial\mathcal{V}'}{\partial w'} = -\frac{G}{r^2}U' - \frac{\mu}{2r^3}w' \end{aligned} \tag{50}$$

in their Hamiltonian form, or

$$\begin{aligned} \ddot{u}' - 2\frac{G}{r^2}\dot{w}' - \frac{G^2+\mu r}{r^4}u' + 2\frac{G\dot{r}}{r^3}w' &= 0, \\ \ddot{w}' + 2\frac{G}{r^2}\dot{u}' - 2\frac{G\dot{r}}{r^3}u' - \frac{G^2-\mu r}{r^4}w' &= 0 \end{aligned} \tag{51}$$

in their Lagrangian form. These are the equations which Brumberg (1961) and Charnyi (1964) considered in preference to the equations (10).

We are able to construct in a direct way the matrizant $R'(t;t_0)$ of the system (50). Let us put

$$K' = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix}, \quad L' = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

because

$$\begin{pmatrix} u \\ w \\ U \\ W \end{pmatrix} = K' \begin{pmatrix} u' \\ w' \\ U' \\ W' \end{pmatrix}, \qquad \begin{pmatrix} u' \\ w' \\ U' \\ W' \end{pmatrix} = L' \begin{pmatrix} u \\ w \\ U \\ W \end{pmatrix},$$

we have that

$$R'(t;t_0) = K' \circ R(t;t_0) \circ L'.$$

This suggests introducing the matrices

$$A'(t) = K' \circ A(t), \qquad B'(t) = B(t) \circ L', \tag{52}$$

so that the matrizant is decomposed into the product

$$R'(t;t_0) = A'(t) \circ B'(t_0). \tag{53}$$

Tables III and IV present the elements of the factor matrices $A'(t)$ and $B'(t)$ which compose the matrizant of the elliptic and hyperbolic Keplerian motions $(G \neq 0,\ e \neq 0,1)$ in the orbital frame of reference.

Table III - The factor $A'(t)$ of the matrizant $R'(t,t_0)$

$a'_{11} = 2r - 3\dot{r}t$	$a'_{12} = 0$
$a'_{21} = -3\frac{G}{r}t$	$a'_{22} = r$
$a'_{31} = -\dot{r} + 3\frac{\mu}{r^2}t$	$a'_{32} = -\frac{G}{r}$
$a'_{41} = -\frac{G}{r}$	$a'_{42} = \dot{r}$
$a'_{13} = G\frac{y}{r}$	$a'_{14} = -G\frac{x}{r}$
$a'_{23} = rY + G\frac{\ }{r}$	$a'_{24} = -rX + G\frac{y}{r}$
$a'_{33} = G\frac{Y}{r}$	$a'_{34} = G\frac{X}{r}$
$a'_{43} = \dot{r}Y - \mu\frac{y}{r^2}$	$a'_{44} = -\dot{r}X + \mu\frac{x}{r^2}$

Table IV - The factor $B'(t)$ in the matrizant $R'(t;t_0)$

$b'_{11} = -\frac{\mu}{2Hr^2}$	$b'_{12} = 0$
$b'_{21} = -\frac{1}{G}(\dot{r} - 3\frac{\mu}{r^2}t)$	$b'_{22} = -\frac{1}{r}$
$b'_{31} = -\frac{\mu}{2HG^2r^2}(Xr^2 - Gy + 3Pt) + \frac{y}{Gr}$	$b'_{32} = \frac{x}{Gr}$
$b'_{41} = -\frac{\mu}{2HG^2r^2}(Yr^2 + Gx + 3Qt) - \frac{x}{Gr}$	$b'_{42} = \frac{y}{Gr}$
$b'_{13} = -\frac{\dot{r}}{2H}$	$b'_{14} = -\frac{G}{2Hr}$
$b'_{23} = \frac{1}{G}(2r - 3\dot{r}t)$	$b'_{24} = -3\frac{t}{r}$
$b'_{33} = \frac{\dot{r}}{2HG^2}(Xr^2 - Gy + 3Pt) + \frac{rx}{G^2}$	$b'_{34} = \frac{1}{2HGr}(Xr^2 - Gy + 3Pt)$
$b'_{43} = -\frac{\dot{r}}{2HG^2}(Yr^2 + Gx + 3Qt) + \frac{ry}{G^2}$	$b'_{44} = -\frac{1}{2HGr}(Yr^2 + Gx + 3Qt)$

6. Hill's equation for the problem of two bodies.

Because we restrict ourselves to elliptic and hyperbolic motions, nowhere along the orbit does the velocity vanish. Hence we define without ambiguity an angle ϕ such that

$$X = V\cos\phi, \qquad Y = V\sin\phi; \tag{54}$$

ϕ is the angle which the velocity vector makes with the inertial axis Ox.

From the definition (54), we immediately draw that

$$\begin{aligned} X\dot{X} + Y\dot{Y} &= V\dot{V}, \qquad & X\ddot{X} + Y\ddot{Y} &= V\ddot{V} - V^2\dot{\phi}^2, \\ X\dot{Y} - \dot{X}Y &= V^2\dot{\phi}, \qquad & X\ddot{Y} - Y\ddot{X} &= 2V\dot{V}\dot{\phi} + V^2\ddot{\phi}. \end{aligned} \tag{55}$$

Now let us for a moment consider a more general dynamic system described by the Hamiltonian function

$$\mathcal{H} = \tfrac{1}{2}(X^2 + Y^2) - \mathcal{U}(x,y). \tag{56}$$

On using the differential equations of motion derived from (56) as well as the identities (55), we obtain that

$$\begin{aligned} \mathcal{U}_x \cos\phi + \mathcal{U}_y \sin\phi &= \dot{V}, \\ \mathcal{U}_x \sin\phi - \mathcal{U}_y \cos\phi &= -V\dot{\phi}, \end{aligned} \tag{57}$$

$$\mathcal{U}_{xx}\cos^2\phi + 2\mathcal{U}_{xy}\cos\phi\sin\phi + \mathcal{U}_{yy}\sin^2\phi = \frac{\ddot{V}}{V} - \dot{\phi}^2,$$

$$\mathcal{U}_{xy}(\cos^2\phi - \sin^2\phi) + (\mathcal{U}_{yy} - \mathcal{U}_{xx})\cos\phi\sin\phi = 2\frac{\dot{V}}{V}\dot{\phi} + \ddot{\phi},$$

$$\mathcal{U}_{xx}\sin^2\phi - 2\mathcal{U}_{xy}\cos\phi\sin\phi + \mathcal{U}_{yy}\cos^2\phi = \mathcal{U}_{xx} + \mathcal{U}_{yy} + \dot{\phi}^2 - \frac{\ddot{V}}{V}.$$

We transform the displacement (u,w,U,W) in the following way:

$$u = u'' \cos\phi - w'' \sin\phi, \qquad U = U'' \cos\phi - W'' \sin\phi,$$
$$w = u'' \sin\phi + w'' \cos\phi, \qquad W = U'' \sin\phi + W'' \cos\phi. \tag{58}$$

The components u'', w'', U'', W'' of the displacement can be interpreted as follows:

(i) u'' is the tangential variation or the variation along the track;

(ii) w'', the normal variation, or the variation across the track;

(iii) U'', the correction in tangential velocity;

(iv) W'', the correction in normal velocity.

The relations (58) define a time-dependent canonical transformation; it can be considered as being generated by the function

$$\mathcal{S}'' \equiv \mathcal{S}''(U,W,u'',w'') = -U(u''\cos\phi + w''\sin\phi) - W(-u''\sin\phi + w''\cos\phi)$$

so that its remainder is the function

$$\frac{\partial \mathcal{S}''}{\partial t} = -(uW - wU)\dot{\phi}.$$

But the variational Hamiltonian associated with (56) is the function

$$\mathcal{V} = \tfrac{1}{2}(U^2 + W^2) - \tfrac{1}{2}(\mathfrak{M}_{xx}u^2 + 2\mathfrak{M}_{xy}uv + \mathfrak{M}_{yy}v^2)$$

which, in virtue of the canonical transformation (58), transforms into the function

$$\mathcal{V}'' \equiv \mathcal{V}''(u'',w'',U'',W'') = \mathcal{V} + \frac{\partial \mathcal{S}''}{\partial t}.$$

In view of the transformation relations (58) and of the identities (57), we readily obtain that

$$\mathcal{V}'' = \tfrac{1}{2}(U''^2 + W''^2) - \dot{\phi}(u''W'' - w''U'')$$
$$-\tfrac{1}{2}\left[\left(\frac{\ddot{V}}{V} - \dot{\phi}^2\right)u''^2 + 2\left(2\frac{\dot{V}}{V}\dot{\phi} + \ddot{\phi}\right)u''w'' + \left(\mathfrak{A}_{xx} + \mathfrak{A}_{yy} + \dot{\phi}^2 - \frac{\ddot{V}}{V}\right)w''^2\right] \tag{59}$$

From the Hamiltonian equations

$$\begin{aligned}
\dot{u}'' &= U'' + \dot{\phi}w'', \\
\dot{w}'' &= W'' - \dot{\phi}u'', \\
\dot{U}'' &= \dot{\phi}W'' + \left(\frac{\ddot{V}}{V} - \dot{\phi}^2\right)u'' + \left(2\frac{\dot{V}}{V}\dot{\phi} + \ddot{\phi}\right)w'', \\
\dot{W}'' &= -\dot{\phi}U'' + \left(2\frac{\dot{V}}{V}\dot{\phi} + \ddot{\phi}\right)u'' + \left(\mathfrak{A}_{xx} + \mathfrak{A}_{yy} + \dot{\phi}^2 - \frac{\ddot{V}}{V}\right)w'',
\end{aligned} \tag{60}$$

we deduce the Lagrangian equation for the normal variations

$$\ddot{w}'' + \left(\frac{\ddot{V}}{V} - 2\dot{\phi}^2 - \mathfrak{A}_{xx} - \mathfrak{A}_{yy}\right)w'' = 2\,\frac{\dot{\phi}}{V}\,(\dot{V}u'' - V\dot{u}''). \tag{61}$$

But the variational integral of energy

$$\delta H = XU + YV - \mathfrak{A}_x u - \mathfrak{A}_y v$$

becomes in the intrinsic variations the function

$$\delta H = VU'' - \dot{V}u'' - V\dot{\phi}w'' \tag{62}$$

or, on using the second of the equations (60),

$$\delta H = (V\dot{u}'' - \dot{V}u'') - 2V\dot{\phi}w'' \tag{63}$$

Therefore the Lagrangian equation (61) transforms into the second order linear differential equation:

$$\ddot{w}'' + \Theta w'' = -2\frac{\dot{\phi}}{V}\delta H \tag{64}$$

whose coefficient is the function

$$\Theta = \frac{\ddot{V}}{V} + 2\dot{\phi}^2 - \mathfrak{M}_{xx} - \mathfrak{M}_{yy}. \tag{65}$$

To conclude, the resolution of the Hamiltonian system (60) reduces to the integration of a second order differential equation in the normal displacement w and the quadrature

$$\frac{d}{dt}\frac{u''}{V} = \frac{1}{V^2}\delta H + 2\frac{\dot{\phi}}{V}w'' \tag{66}$$

as indicated by the variational integral (63) for the tangential displacement.

We now return to the problem of two bodies. In this case, in view of the identities (55) and of the orbital equations of motion, we find that

$$\dot{V} = -\frac{\mu\dot{r}}{Vr^2}, \qquad \dot{\phi} = \frac{\mu G}{V^2 r^3}$$

$$\ddot{V} = \frac{\mu}{Vr^3}\left[2\dot{r}^2 - \frac{G^2}{V^2 r^2}\left(V^2 - \frac{\mu}{r}\right)\right]$$

$$\mathfrak{M}_{xx} + \mathfrak{M}_{yy} = \frac{\mu}{r^3}$$

Accordingly, the coefficient (65) in Hill's equation (64) is the function

$$\Theta = \frac{\mu}{V^2 r^3}\left[V^2 - 3\frac{G^2}{V^2 r^2}\left(V^2 - \frac{\mu}{r}\right)\right] \tag{67}$$

which can also be written as

$$\Theta = \frac{\mu}{V^2 r^3}\left[V^2 + 3G(\dot{\phi} - \dot{\theta})\right]$$

to provide a better indication of its physical meaning.

7. The matrizant in the intrinsic frame of reference.

In the same way as we did in the orbital frame of reference, we decompose the matrizant $R''(t;t_0)$ in the intrinsic frame of reference into the product

$$R''(t;t_0) = A''(t)\circ B''(t_0),$$

with

$$A''(t) = K''\circ A(t), \qquad B''(t) = B(t)\circ L'',$$

$$K'' = \begin{pmatrix} \cos\phi & \sin\phi & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & -\sin\phi & \cos\phi \end{pmatrix}, \qquad L'' = \begin{pmatrix} \cos\phi & \sin\phi & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & -\sin\phi & \cos\phi \end{pmatrix}$$

The elements of the matrices $A''(t)$ and $B''(t)$ are listed in Tables V and VI.

We denote by $\underset{\sim}{a}''_i$ the vector whose components are the elements of the i^{th} column in the matrix $A(t)$. For $1 \leq i \leq 4$, the vector $\underset{\sim}{a}''_i$ is a solution of the variational equations. Let us compute the value taken by the variational integral of energy along that variation; we find that

$$(\nabla H|\underset{\sim}{a}''_1) = -2H, \qquad \text{and} \qquad (\nabla H|\underset{\sim}{a}''_j) = 0 \qquad (2 \leq j \leq 4). \tag{68}$$

Variations $\underset{\sim}{u}$ such that $(\nabla H|\underset{\sim}{u}) = 0$ are called isoenergetic. Thus what the equalities (68) prove is that the isoenergetic variations constitute a three-dimensional space, and that the variations a''_j $(j = 2,3,4)$ form a base of that vector subspace.

Table V - The factor $B''(t)$ in the matrizant $R''(t;t_0)$

$a''_{11} = 2\frac{r\dot{r}}{V} - 3Vt$	$a''_{12} = \frac{G}{V}$
$a''_{21} = -2\frac{G}{V}$	$a''_{22} = \frac{r\dot{r}}{V}$
$a''_{31} = -V - 3\dot{V}t$	$a''_{32} = 0$
$a''_{41} = -3V\dot{\phi}t$	$a''_{42} = V$
$a''_{13} = 2G\frac{Y}{V}$	$a''_{14} = -2G\frac{X}{V}$
$a''_{23} = \frac{1}{V}(2GX + V^2y)$	$a''_{24} = \frac{1}{V}(2GY - V^2x)$
$a''_{33} = -Vy\dot{\phi}$	$a''_{34} = Vx\dot{\phi}$
$a''_{43} = VY + \dot{V}y$	$a''_{44} = -VX - \dot{V}x$

Table VI - The factor $B''(t)$ in the matrizant $R''(t;t_0)$

$$b''_{11} = \frac{\dot V}{2H} \qquad b''_{12} = \frac{V\dot\phi}{2H}$$

$$b''_{21} = -\frac{V}{G} - 3\frac{\dot V}{G}t \qquad b''_{22} = -3\frac{V\dot\phi}{G}t$$

$$b''_{31} = \frac{\dot V}{2HG^2}(Xr^2 - Gy + 3Pt) + \frac{Y}{GV} \qquad b''_{32} = \frac{V\dot\phi}{2HG^2}(Xr^2 - Gy + 3Pt) + \frac{GX}{V}$$

$$b''_{41} = \frac{\dot V}{2HG^2}(Yr^2 + Gx + 3Qt) - \frac{X}{GV} \qquad b''_{42} = \frac{V\dot\phi}{2HG^2}(Yr^2 + Gx + 3Qt) + \frac{GY}{V}$$

$$b''_{13} = -\frac{V}{2H} \qquad b''_{14} = 0$$

$$b''_{23} = \frac{2r\dot r}{GV} - 3\frac{V}{G}t \qquad b''_{24} = -\frac{2}{V}$$

$$b''_{33} = \frac{V}{2HG^2}(Xr^2 - Gy + 3Pt) + \frac{xr\dot r}{VG^2} \qquad b''_{34} = -\frac{x}{GV}$$

$$b''_{43} = -\frac{V}{2HG^2}(Yr^2 + Gx + 3Qt) + \frac{yr\dot r}{VG^2} \qquad b''_{44} = -\frac{y}{GV}$$

Foremost among the isoenergetic variations is the solution $\underset{\sim}{u}_0 = (u_0, w_0, U_0, W_0)$ such that

$$u_0 = X, \qquad w_0 = Y, \qquad U_0 = \dot{X}, \qquad V_0 = \dot{Y};$$

its components in the intrinsic frame of reference are

$$u_0'' = V, \qquad w_0'' = 0, \qquad U_0'' = \dot{V}, \qquad W_0'' = -V\dot{\phi} \tag{70}$$

The reader could check that this remarkable isoenergetic displacement is the linear combination

$$\underset{\sim}{u}_0 = \alpha_0 \underset{\sim}{a}_2'' + \beta_0 \underset{\sim}{a}_3'' + \gamma_0 \underset{\sim}{a}_3''$$

whose coefficients are the functions

$$\alpha_0 = -\,4\dot{\phi} - 2\frac{H}{G},$$

$$\beta_0 = \frac{x}{G^2}\left(V^2 - \frac{\mu}{r}\right) + 2\frac{x}{G}\dot{\phi} + \frac{\dot{V}X}{VG\dot{\phi}}\ ,$$

$$\gamma_0 = \frac{y}{G^2}\left(V^2 - \frac{\mu}{r}\right) + 2\frac{y}{G}\dot{\phi} + \frac{\dot{V}Y}{VG\dot{\phi}}\ .$$

One usually completes the base of the vector subspace of isoenergetic variations by adjoining to the vector (70) the solutions $\underset{\sim}{u}_1$ and $\underset{\sim}{u}_2$ defined by the following initial conditions:

$$\begin{aligned} &u_1''(t_0) = 0, \qquad w_1''(t_0) = 1, \qquad \dot{w}_1''(t_0) = 0, \\ &u_2''(t_0) = 0, \qquad w_2''(t_0) = 0, \qquad \dot{w}_2''(t_0) = 1. \end{aligned} \tag{71}$$

The variation $\underset{\sim}{u}_1$ results from displacing the initial position along the normal to the orbit at that point without modifying the velocity, while the

variation $\underset{\sim}{u}_2$ is caused by an initial change of the normal component of the velocity (or for that matter an initial impulse in the normal direction) without changing the initial position. To make such displacements isoenergetic, we adjust the change in the tangential component of the velocity according to the variational integral of energy (63) in which we put $\delta H = 0$. There results that

$$\dot{u}_1''(t_0) = 2\dot{\phi}(t_0), \qquad \dot{u}_2''(t_0) = 0. \tag{72}$$

Since we assume that they are isoenergetic, the variations $\underset{\sim}{u}_1$ and $\underset{\sim}{u}_2$ are linear combinations of the kind

$$\underset{\sim}{u}_1 = \alpha_1\underset{\sim}{a}_2'' + \beta_1\underset{\sim}{a}_3'' + \gamma_1\underset{\sim}{a}_4'',$$

$$\underset{\sim}{u}_2 = \alpha_2\underset{\sim}{a}_2'' + \beta_2\underset{\sim}{a}_3'' + \gamma_2\underset{\sim}{a}_4'' .$$

They satisfy the initial conditions (71) if and only if

$$\alpha_1 = 2\frac{\dot{V}_0}{V_0^2}, \qquad \beta_1 = \frac{V_0X_0-\dot{V}_0x_0}{GV_0^2}, \qquad \gamma_1 = \frac{V_0Y_0-\dot{V}_0y_0}{GV_0^2}$$

$$\alpha_2 = \frac{2}{V_0} \qquad \beta_2 = -\frac{x_0}{GV_0} \qquad \gamma_2 = -\frac{y_0}{GV_0}.$$

Accordingly their components are as follows

$$u_1'' = \frac{2}{V_0^2V}\left[G\dot{V}_0 + (V_0X_0-\dot{V}_0x_0)Y - (V_0Y_0-\dot{V}_0y_0)X\right],$$

$$w_1'' = \frac{1}{GV_0^2V}\left[2G\dot{V}_0r\dot{r} + (V_0X_0-\dot{V}_0x_0)(2GX+V_y^2) + (V_0Y_0-\dot{V}_0y_0)(2GY-V^2x)\right],$$

$$U_1'' = \frac{V\dot{\phi}}{GV_0{}^2}\left[(V_0Y_0-\dot{V}_0y_0)x - (V_0X_0-\dot{V}_0x_0)y\right],$$

$$W_1'' = \frac{1}{GV_0^2}\left[2G\dot{V}_0V + (V_0X_0-\dot{V}_0x_0)(VY+\dot{V}y) - (V_0Y_0-\dot{V}_0y_0)(VX+\dot{V}x)\right] \tag{73}$$

for the basic isoenergetic displacement $\underset{\sim}{u}_1$, and

$$\begin{aligned} u_2'' &= \frac{2}{V_0V}(G - Yx_0 + Xy_0), \\ w_2'' &= \frac{1}{GV_0V}\left[2Gr\dot{r} - (V^2y+2GX)x_0 + (V^2x-2GY)y_0\right], \\ U_2'' &= \frac{V\dot{\phi}}{GV_0}(yx_0 - xy_0), \\ W_2'' &= \frac{1}{GV_0}\left[2GV - (VY+\dot{V}y)x_0 + (VX+\dot{V}x)y_0\right] \end{aligned} \tag{74}$$

for the other basic isoenergetic displacement $\underset{\sim}{u}_2$.

Conclusion

The variational equations of the problem of two bodies have been solved from first principles applying Jacobi's dual theorem of the last multiplier to the adjoint of the Hamiltonian system expressed in Cartesian coordinates with respect to an arbitrarily given inertial frame of reference. Once the matrizant $R(t;t_0)$ in that system has been obtained in closed form, other forms of the resolvent are derived simply by homogeneous canonical extensions. By way of illustration, we have recovered in this way the matrizants in the orbital frame of reference and in Hill's intrinsic coordinate system. To our knowledge this last resolvent is not to be found elsewhere in the literature on the problem of two bodies.

The treatment we give in this paper is incomplete in many respects. For instance some who deal with orbital transfers or trajectory designs by matching conics would like to introduce a so-called *universal* variable which, for Keplerian motions as functions of the eccentricity e, acts as a locally uniformizing variable (Sconzo 1967) in the neighborhood of $e = 1$. Goodyear (1965) has been the first to introduce a universal time variable in the matrizant of the problem of two bodies; he did it as an extension of the method proposed by Stumpff (1947, 1959, 1962) for the orbital equations. At this point, the problem is to find a way of simplifying the lengthy and, at places, cumbersome manipulations performed by Goodyear (1966).

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